

Exponential sums with coefficients of certain Dirichlet series

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Abstract

Under the generalized Lindelöf Hypothesis in the t - and q -aspects, we bound exponential sums with coefficients of Dirichlet series belonging to a certain class. We use these estimates to establish a conditional result on squares of Hecke eigenvalues at Piatetski-Shapiro primes.

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1 General assumptions

In this paper, we derive a conditional estimate for exponential sums of the form

$$\sum_{n \sim N} a_n e(f(n)),$$

where a_n is the n -th coefficient of Dirichlet series $F(s)$ whose twists with Dirichlet characters satisfy the generalized Lindelöf Hypothesis in the t - and q -aspects, and $f(x)$ is a function having certain properties. As an application, we consider squares of Hecke eigenvalues at Piatetski-Shapiro primes. In the following, we state the required conditions on $F(s)$ and $f(x)$.

Conditions on the L -function:

We assume that

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is a Dirichlet series, absolutely convergent for $\Re s > 1$, which satisfies the following conditions a), b) and c).

a) $F(s)$ lies in the extended Selberg class of Dirichlet series which don't necessarily possess a functional equation, i.e. $F(s)$ has the following properties.

(i) (*Analiticity*) There exists some $m \in \mathbb{N}$ such that $(s-1)^m F(s)$ extends to an entire function of finite order.

(ii) (*Ramanujan conjecture*) $a_1 = 1$ and $a_n \ll_{\varepsilon} n^{\varepsilon}$ for any $\varepsilon > 0$.

(iii) (*Euler product*) For $\Re s > 1$, the function $F(s)$ can be written as a product over primes in the form

$$F(s) = \prod_p F_p(s),$$

where $\log F_p(s)$ is a Dirichlet series of the form

$$\log F_p(s) = \sum_{n=0}^{\infty} b_{p^k} p^{-ks}$$

with complex coefficients b_{p^k} satisfying

$$b_{p^k} = O(p^{k\theta})$$

for some $\theta < 1/2$.

b) For any Dirichlet character χ define

$$F(s, \chi) := \sum_{n=1}^{\infty} a_n \chi(n) n^{-s} \quad \text{for } \Re s > 1.$$

Then $(s-1)^m F(s, \chi)$ extends to an entire function again.

c) The family of functions $F(s, \chi)$ satisfies the Lindelöf Hypothesis in the t - and q -aspects, i.e.

$$F\left(\frac{1}{2} + it, \chi\right) \ll |tq|^\varepsilon \quad \text{for all } |t| \geq 1, q \in \mathbb{N} \text{ and characters } \chi \bmod q. \quad (1.1)$$

Conditions on f :

We assume that $f : [1, \infty) \rightarrow \mathbb{R}$ satisfies the following conditions a)-f).

- a) f is three times continuously differentiable.
- b) f is monotonically increasing.
- c) $f(x) \asymp f(2x)$ for all $x \geq 1$.
- d) $f^{(k)}(x) \asymp f(x)/x^k$ for all $x \geq 1$ and $k = 1, 2, 3$.
- e) $f'(x) + xf''(x) \asymp f(x)/x$ for all $x \geq 1$.
- f) $2f''(x) + xf'''(x) \asymp f(x)/x^2$ for all $x \geq 1$.

2 Results

Our first result is the following.

Theorem 1. *Fix $\eta > 0$. Suppose that $1 \leq N < N' \leq 2N$ and*

$$N^{1/2+\eta} \leq f(N) \leq N^{3/2-\eta}. \quad (2.1)$$

Then, under the conditions in section 1, we have

$$\sum_{N < n \leq N'} a_n e(f(n)) \ll_{f, \eta, \varepsilon} N^{19/22+\varepsilon} f(N)^{1/11}.$$

We note that the above bounds are non-trivial if $\varepsilon < \eta/11$ and N is large enough.

With applications in mind, we also prove the following modification of Theorem 1.

Theorem 2. *Let $m \in \mathbb{N}$. Then, under the conditions in Theorem 1 and section 1, we have*

$$\sum_{\substack{n \sim N \\ (n,m)=1}} \mu^2(n) a_n e(f(n)) \ll_{f,\eta,\varepsilon} m^\varepsilon N^{19/22+\varepsilon} f(N)^{1/11}.$$

Let now G be a Hecke eigenform of weight κ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. By $\lambda(n)$ we denote the normalized n -th Fourier coefficient of G , *i.e.*

$$G(z) = \sum_{n=1}^{\infty} \lambda(n) n^{(\kappa-1)/2} e(nz) \quad \text{for } \Im z > 0, \text{ and } \lambda(1) = 1.$$

These Hecke eigenvalues satisfy the multiplicative property

$$\lambda(mn) = \sum_{d|\gcd(m,n)} \mu(d) \lambda\left(\frac{m}{d}\right) \lambda\left(\frac{n}{d}\right) \quad \text{for all } m, n \in \mathbb{N} \quad (2.2)$$

and the Ramanujan conjecture $\lambda(n) \ll_\varepsilon n^\varepsilon$.

Let $L(\mathrm{Sym}^2 G, s)$ be the symmetric square L -function for G , defined by

$$L(\mathrm{Sym}^2 G, s) = \zeta(2s) \sum_{n=1}^{\infty} \lambda(n^2) n^{-s} \quad \text{for } \Re s > 1.$$

We note that by multiplying out the right-hand side, we get

$$L(\mathrm{Sym}^2 G, s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{for } \Re s > 1, \quad (2.3)$$

where the coefficients of the Dirichlet series on the right-hand side satisfy $a_n = \lambda(n^2)$ for any squarefree n . Moreover, it is well-known that $L(\mathrm{Sym}^2 G, s)$ lies in the Selberg class and hence satisfies condition a) in section 1.

More generally, for any Dirichlet character χ let $L(\mathrm{Sym}^2 G \otimes \chi, s)$ be the symmetric square L -function for G twisted with χ , defined by

$$L(\mathrm{Sym}^2 G \otimes \chi, s) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s} = L(2s, \chi^2) \sum_{n=1}^{\infty} \chi(n) \lambda(n^2) n^{-s} \quad \text{for } \Re s > 1.$$

As a consequence of the work of Shimura [5], $L(\mathrm{Sym}^2 G \otimes \chi, s)$ extends analytically to the whole complex plane and hence satisfies condition b) in section 1. If χ is primitive, then $L(\mathrm{Sym}^2 G \otimes \chi, s)$ even lies in the Selberg class.

The Lindelöf Hypothesis in the t - and q -aspects for the family of L -functions $L(\mathrm{Sym}^2 G \otimes \chi, s)$, with G fixed, asserts that

$$L\left(\mathrm{Sym}^2 G \otimes \chi, \frac{1}{2} + it\right) \ll (tq)^\varepsilon \quad \text{for all } |t| \geq 1, q \in \mathbb{N} \text{ and characters } \chi \bmod q. \quad (2.4)$$

We note that it can be deduced from Theorem 1 in [2] that (2.4) holds if $L(\mathrm{Sym}^2 G \otimes \chi, s)$ satisfies the Riemann Hypothesis for all primitive characters χ .

In [1], we bounded the average of $\lambda(p)$ at Piatetski-Shapiro primes, *i.e.* primes of the form $p = [n^c]$ with $n \in \mathbb{N}$ and $c > 1$ fixed. The c -range for which we obtained a non-trivial result was $1 < c < 8/7$. In this range, we proved that

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda([n^c]) \ll N \exp\left(-C\sqrt{\log N}\right), \quad (2.5)$$

where here as in the sequel, \mathbb{P} is the set of primes. We posed the question if also an asymptotic estimate for the average of the *squares* of these Hecke eigenvalues at Piatetski-Shapiro can be established. Employing Theorem 2, we shall prove the following conditional result.

Theorem 3. *Let $1 < c < 25/24$ be fixed and \mathbb{P} be the set of primes. Assume that (2.4) holds. Then we have*

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda([n^c])^2 \sim \frac{N}{c \log N} \quad \text{as } N \rightarrow \infty.$$

According to [1], Theorem 3 and (2.5) imply the following result on the sign changes of $\lambda(p)$ at Piatetski-Shapiro primes p .

Theorem 4. *Let $1 < c < 25/24$ be fixed and assume that (2.4) holds. Then $\lambda(p)$ changes sign infinitely often as p runs through the primes of the form $p = [n^c]$ with $n \in \mathbb{N}$.*

We point out that the full strength of the Lindelöf hypothesis is not required to obtain non-trivial bounds for the exponential sums in question. However, in this paper, we want to establish the strongest possible result that our method allows.

3 Farey dissection

Our goal is to establish a non-trivial bound for the exponential sum

$$\sum_{n \sim N} a_n e(f(n))$$

in Theorem 1. To this end, we shall split this exponential sum into short subsums using a Farey dissection of a certain interval. We note that the splitting of the summation interval in the present paper differs from that in [1]. It will become clear in the next section why it is advantageous to split the summation interval as described below.

For $x \geq 1$, let

$$h(x) := f'(x) + x f''(x). \quad (3.1)$$

By the condition f) on f in section 1, we have

$$h'(x) = 2f''(x) + x f'''(x) \asymp \frac{f(x)}{x^2}. \quad (3.2)$$

Hence, $h(x)$ is monotonically increasing or decreasing. In the sequel, we assume without loss of generality that $h(x)$ is monotonically decreasing (in particular, if $f(x)$ is defined as in (10.7) in section 10, then $h(x)$ will have this property). Let Q be a real parameter with

$$1 \leq Q \leq N, \quad (3.3)$$

to be chosen later.

Now we make a Farey dissection of level Q of the interval $[h(N'), h(N)]$ (for details on Farey intervals, see [1], for example). In this way, we write $[h(N'), h(N)]$ as the disjoint union of intervals of the form

$$\left[\frac{l}{q} - \frac{M_1}{qQ}, \frac{l}{q} + \frac{M_2}{qQ} \right) \cap [h(N'), h(N)],$$

where $M_1, M_2 \asymp 1$, $q \leq Q$ and $(q, l) = 1$. Projecting these intervals back into $(N, N']$ under the map h^{-1} , we get intervals of the form

$$h^{-1} \left(\left[\frac{l}{q} - \frac{M_1}{qQ}, \frac{l}{q} + \frac{M_2}{qQ} \right] \cap [h(N'), h(N)) \right) = (x_0 - m_1, x_0 + m_2] \subseteq (N, N']$$

with

$$x_0 = h^{-1} \left(\frac{l}{q} \right)$$

and

$$m_1, m_2 \ll \frac{1}{qQ} \cdot (h^{-1})' \left(\frac{l}{q} \right) = \frac{1}{qQ} \cdot \frac{1}{h'(x_0)} \asymp \frac{N^2}{qQf(N)},$$

by (3.2) and the conditions b) and c) on f in section 1.

In the following sections, we shall estimate the subsums

$$\sum_{x_0 - m_1 < n \leq x_0 + m_2} a_n e(f(n)).$$

4 Approximation of f and partial summation

In $(x_0 - m_1, x_0 + m_2]$, we now approximate the function $f(x)$ by

$$g(x) = h(x_0)x - x_0^2 f''(x_0) \log x + C = \frac{l}{q} \cdot x - x_0^2 f''(x_0) \log x + C, \quad (4.1)$$

where

$$C := f(x_0) - h(x_0)x_0 + x_0^2 f''(x_0) \log x_0.$$

Using the definition of $h(x)$ in (3.1), It follows that

$$g(x_0) = f(x_0), \quad g'(x_0) = f'(x_0), \quad \text{and} \quad g''(x_0) = f''(x_0).$$

Hence, applying Taylor's theorem to approximate $(f - g)'(x)$ near x_0 , we have

$$f'(x) - g'(x) = \frac{1}{2}(x - x_0)^2 (f'''(c) - g'''(c))$$

for some $c \in [x_0 - m_1, x_0 + m_2]$ if $x \in (x_0 - m_1, x_0 + m_2]$. Now,

$$f'''(c) - g'''(c) = f'''(c) - \frac{2x_0^2 f''(x_0)}{c^3} \ll \frac{f(N)}{N^3},$$

by our conditions on f . Hence,

$$f'(x) - g'(x) \ll \frac{N^4}{q^2 Q^2 f(N)^2} \cdot \frac{f(N)}{N^3} \ll \frac{N}{q^2 Q^2 f(N)}.$$

Using partial summation, we deduce that

$$\begin{aligned}
& \sum_{n \in (x_0 - m_1, x_0 + m_2]} a_n e(f(n)) \\
&= \sum_{n \in (x_0 - m_1, x_0 + m_2]} a_n e(g(n)) e(f(n) - g(n)) \\
&= e(f(x_0 + m_2) - g(x_0 + m_2)) \sum_{n \in (x_0 - m_1, x_0 + m_2]} a_n e(g(n)) - \\
& \quad 2\pi i \int_{x_0 - m_1}^{x_0 + m_2} \left(\sum_{n \in (x_0 - m_1, u]} a_n e(g(n)) \right) (f'(u) - g'(u)) e(f(u) - g(u)) du \\
&\ll \left(1 + (m_1 + m_2) \cdot \frac{N}{q^2 Q^2 f(N)} \right) \cdot \max_{u \leq x_0 + m_2} \left| \sum_{n \in (x_0 - m_1, u]} a_n e(g(n)) \right| \\
&\ll \left(1 + \frac{N^3}{q^3 Q^3 f(N)^2} \right) \cdot \max_{u \leq x_0 + m_2} \left| \sum_{n \in (x_0 - m_1, u]} a_n e(g(n)) \right|.
\end{aligned} \tag{4.2}$$

Thus we have replaced the function $f(x)$ by $g(x)$. The exponential sum with $g(n)$ in place of $f(n)$ can now be related to the functions $F(s, \chi)$. This will be done in the next sections.

In [1], we approximated the function $f(x)$ just by a linear function of the form

$$g(x) = \frac{l}{q} \cdot x + C$$

in an interval around the point $x_0 = f'^{-1}(l/q)$. However, in this way one can just force the first derivative of $g(x)$ to agree with that of $f(x)$ at the point at $x = x_0$. The approximation of $f(x)$ by the function $g(x)$ defined in (4.1) allows to force the first *and the second* derivatives of $g(x)$ and $f(x)$ to agree at $x = x_0$. This reduces the error in the approximation substantially and is the key point of this paper.

5 Rewriting $\sum_n a_n e(g(n))$ using multiplicative characters

We have

$$\sum_{x_0 - m_1 < n \leq u} a_n e(g(n)) = \sum_{x_0 - m_1 < n \leq u} a_n e\left(n \cdot \frac{l}{q}\right) \cdot n^{-iT} \tag{5.1}$$

with

$$T := 2\pi x_0^2 f''(x_0). \tag{5.2}$$

We break the sum over n as follows.

$$\begin{aligned}
\sum_{x_0 - m_1 < n \leq u} a_n e\left(n \cdot \frac{l}{q}\right) \cdot n^{-iT} &= \sum_{d|q} \sum_{\substack{x_0 - m_1 < n \leq u \\ (n, q) = d}} a_n e\left(n \cdot \frac{l}{q}\right) \cdot n^{-iT} \\
&= \sum_{d|q} d^{-iT} \sum_{\substack{(x_0 - m_1)/d < n \leq u/d \\ (n, q/d) = 1}} a_{dn} e\left(n \cdot \frac{l}{q/d}\right) \cdot n^{-iT}.
\end{aligned}$$

Now we write the additive character in the last line using multiplicative characters in the form

$$e\left(n \cdot \frac{l}{q/d}\right) = \frac{1}{\varphi(q/d)} \cdot \sum_{\chi \bmod q/d} \bar{\chi}(l) \tau(\bar{\chi}) \chi(n).$$

It follows that

$$\begin{aligned} & \sum_{x_0 - m_1 \leq n \leq u} a_n e\left(n \cdot \frac{l}{q}\right) \cdot n^{-iT} \\ &= \sum_{d|q} d^{-iT} \sum_{\chi \bmod q/d} \frac{1}{\varphi(q/d)} \cdot \bar{\chi}(l) \tau(\bar{\chi}) \sum_{(x_0 - m_1)/d < n \leq u/d} a_{dn} \chi(n) n^{-iT}. \end{aligned} \quad (5.3)$$

6 Reduction to $F(s, \chi)$

Using Perron's formula and the Ramanujan conjecture, $a_n \ll n^\varepsilon$, we have

$$\begin{aligned} \sum_{(x_0 - m_1)/d < n \leq u/d} a_{dn} \chi(n) n^{-iT} &= \int_{c-iT_0}^{c+iT_0} \left(\sum_{n=1}^{\infty} a_{dn} \chi(n) n^{-s-iT} \right) \left(\left(\frac{u}{d} \right)^s - \left(\frac{x_0 - m_1}{d} \right)^s \right) \frac{ds}{s} + \\ &O\left(\frac{N^{1+\varepsilon}}{dT_0} + N^\varepsilon \right) \end{aligned} \quad (6.1)$$

for $c = 1 + 1/\log N$ and $T_0 \geq 1$, where we recall that

$$N \leq x_0 - m_1 < u \leq N' \leq 2N. \quad (6.2)$$

Next, we relate the Dirichlet series in the integrand to $F(s, \chi)$.

Since $F(s)$ has an Euler product, the coefficients a_n of $F(s)$ are multiplicative in n . Hence, for $\Re s > 1$, we have

$$\sum_{n=1}^{\infty} a_{dn} \chi(n) n^{-s} = \left(\sum_{\substack{n=1 \\ s(n)|d}}^{\infty} a_{dn} \chi(n) n^{-s} \right) \cdot \left(\sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} a_{dn} \chi(n) n^{-s} \right),$$

where $s(n)$ is the largest squarefree number dividing n , and we may write

$$\sum_{\substack{n=1 \\ s(n)|d}}^{\infty} a_{dn} \chi(n) n^{-s} = \prod_{p|d} \sum_{k=0}^{\infty} a_{p^{\alpha(p)+k}} \chi^k(p) p^{-ks},$$

where

$$d = \prod_{p|d} p^{\alpha(p)}$$

is the prime number factorization of d . Further,

$$\sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} a_n \chi(n) n^{-s} = F(s, \chi) \prod_{p|d} \left(\sum_{k=0}^{\infty} a_{p^k} \chi^k(p) p^{-ks} \right)^{-1}.$$

So altogether,

$$\sum_{n=1}^{\infty} a_{dn} \chi(n) n^{-s} = G_d(s, \chi) F(s, \chi)$$

with

$$G_d(s, \chi) = \prod_{p|d} \frac{\sum_{k=0}^{\infty} a_{p^{\alpha(p)+k}} \chi^k(p) p^{-ks}}{\sum_{k=0}^{\infty} a_{p^k} \chi^k(p) p^{-ks}}.$$

Hence, the integral on the right-hand side of (6.1) takes the form

$$\begin{aligned} & \int_{c-iT_0}^{c+iT_0} \left(\sum_{n=1}^{\infty} a_{dn} \chi(n) n^{-s-iT} \right) \left(\left(\frac{u}{d} \right)^s - \left(\frac{x_0 - m_1}{d} \right)^s \right) \frac{ds}{s} \\ &= \int_{c-iT_0}^{c+iT_0} G_d(s+iT, \chi) F(s+iT, \chi) \left(\left(\frac{u}{d} \right)^s - \left(\frac{x_0 - m_1}{d} \right)^s \right) \frac{ds}{s}. \end{aligned} \tag{6.3}$$

7 Estimation of the integral

We shall need a bound for $G_d(s, \chi)$ if $\Re s \geq 1/2$, which we establish in the following. By the Ramanujan conjecture, $a_n \ll n^\varepsilon$, we have

$$\sum_{k=0}^{\infty} a_{p^{\alpha(p)+k}} \chi^k(p) p^{-ks} \ll p^{\alpha(p)\varepsilon} \quad \text{uniformly for } \Re s \geq \frac{1}{2}.$$

Let $t_0 \in \mathbb{R}$ such that

$$p^{-it_0} = \chi(p).$$

Then

$$\sum_{k=0}^{\infty} a_{p^k} \chi^k(p) p^{-ks} = \sum_{k=0}^{\infty} a_{p^k} p^{-k(s+it_0)},$$

and by axiom (iii) (Euler product) for $F(s)$, we therefore have

$$\log \sum_{k=0}^{\infty} a_{p^k} \chi^k(p) p^{-ks} = \log F_p(s+it_0) = \sum_{k=0}^{\infty} b_{p^k} p^{-k(s+it_0)} = \sum_{k=0}^{\infty} b_{p^k} \chi^k(p) p^{-ks},$$

where b_{p^k} ($k = 0, 1, \dots$) are suitable coefficients satisfying

$$b_{p^k} \ll p^{k\theta}$$

for some $\theta < 1/2$. It follows that

$$\left(\sum_{k=0}^{\infty} a_{p^k} \chi^k(p) p^{-ks} \right)^{-1} = O(1) \quad \text{uniformly for } \Re s \geq 1/2. \tag{7.1}$$

From the above estimates, we deduce that

$$G_d(s, \chi) \ll d^\varepsilon \quad \text{uniformly for } \Re s \geq 1/2.$$

Furthermore, using conditions b) and c) on $F(s, \chi)$ in section 1 together with the Phragmen-Lindelöf principle, we have

$$F(\sigma + it, \chi) \ll (tq)^\varepsilon \quad \text{uniformly for } \sigma \geq \frac{1}{2} \text{ and } |\sigma + it - 1| \geq \frac{1}{4}.$$

Thus,

$$G_d(\sigma + it, \chi)F(\sigma + it, \chi) \ll (dtq)^\varepsilon \quad \text{uniformly for } \sigma \geq \frac{1}{2} \text{ and } |\sigma + it - 1| \geq \frac{1}{4}. \quad (7.2)$$

Now we bound the integral on the right-hand side of (6.3), where we suppose that

$$T_0 \leq \frac{T}{2}. \quad (7.3)$$

Using Cauchy's integral theorem, we then have

$$\begin{aligned} & \int_{c-iT_0}^{c+iT_0} G_d(s+iT, \chi)F(s+iT, \chi) \left(\left(\frac{u}{d} \right)^s - \left(\frac{x_0 - m_1}{d} \right)^s \right) \frac{ds}{s} \\ &= \left(\int_{c-iT_0}^{1/2-iT_0} + \int_{1/2-iT_0}^{1/2+iT_0} + \int_{1/2+iT_0}^{c+iT_0} \right) G_d(s+iT, \chi)F(s+iT, \chi) \times \\ & \quad \left(\left(\frac{u}{d} \right)^s - \left(\frac{x_0 - m_1}{d} \right)^s \right) \frac{ds}{s}. \end{aligned} \quad (7.4)$$

From (6.2), (7.2), (7.3), (7.4) and $d|q$, we deduce that

$$\int_{c-iT_0}^{c+iT_0} G_d(s+iT, \chi)F(s+iT, \chi) \left(\left(\frac{u}{d} \right)^s - \left(\frac{x_0 - m_1}{d} \right)^s \right) \frac{ds}{s} \ll (Tq)^\varepsilon \cdot \left(\left(\frac{N}{d} \right)^{1/2} + \frac{N}{dT_0} \right) \quad (7.5)$$

if $T \geq 1/2$.

8 Proof of Theorem 1

Now we choose

$$T_0 := \left(\frac{N}{d} \right)^{1/2}. \quad (8.1)$$

We note that

$$T \asymp f(N) \quad (8.2)$$

by (5.2), $x_0 \asymp N$ and our conditions of f . Hence, by (2.1), the condition (7.3) is satisfied if N is large enough. From (2.1), (6.1), (6.3), (7.5), (8.1), (8.2) and $d \leq N$ (by $d|q$, $q \leq Q$ and (3.3)), we deduce that

$$\sum_{(x_0 - m_1)/d < n \leq u/d} a_{dn} \chi(n) n^{-iT} \ll (qN)^\varepsilon \cdot \left(\frac{N}{d} \right)^{1/2}.$$

Plugging this into (5.3), and using $\varphi(q/d) \gg q^{1-\varepsilon}/d$ and $|\tau(\bar{\chi})| \leq \sqrt{q/d}$, we obtain

$$\left| \sum_{x_0 - m_1 < n \leq u} a_n e \left(n \cdot \frac{l}{q} \right) n^{-iT} \right| \ll (qN)^{1/2+\varepsilon}. \quad (8.3)$$

This together with (4.2) and (5.1) yields

$$\sum_{n \in (x_0 - m_1, x_0 + m_2]} a_n e(f(n)) \ll \left(1 + \frac{N^3}{q^3 Q^3 f(N)^2}\right) (qN)^{1/2+\varepsilon}. \quad (8.4)$$

In section 3, we have divided the interval $[h(N'), h(N))$ into Farey intervals around fractions l/q with

$$1 \leq q \leq Q, \quad l \asymp q \cdot h(N) \asymp q \cdot \frac{f(N)}{N} \quad \text{and} \quad (q, l) = 1.$$

Hence, summing the contributions of the short sums in (8.4) over all relevant q and l , we get

$$\begin{aligned} \sum_{n \sim N} a_n e(f(n)) &\ll \sum_{q \leq Q} \sum_{l \asymp q f(N)/N} \left(1 + \frac{N^3}{q^3 Q^3 f(N)^2}\right) (qN)^{1/2+\varepsilon} \\ &\ll \left(\frac{Q^{5/2} f(N)}{N^{1/2}} + \frac{N^{5/2}}{Q^3 f(N)}\right) (QN)^\varepsilon. \end{aligned}$$

Now we choose

$$Q := \left(\frac{N^{5/2}}{f(N)} \cdot \frac{N^{1/2}}{f(N)}\right)^{2/11} = \frac{N^{6/11}}{f(N)^{4/11}}.$$

Thus we get

$$\sum_{n \sim N} a_n e(f(n)) \ll N^{19/22+\varepsilon} f(N)^{1/11},$$

which completes the proof. \square

9 Proof of Theorem 2

Theorem 2 can be proved along similar lines as Theorem 1. The arguments in sections 3-5 carry over completely. We are then led to the sum

$$\sum_{\substack{(x_0 - m_1)/d < n \leq t/d \\ (n, m) = 1}} \mu^2(dn) a_{dn} \chi(n) n^{-iT}$$

in place of the sum

$$\sum_{(x_0 - m_1)/d < n \leq t/d} a_{dn} \chi(n) n^{-iT}$$

considered in sections 6-8. We now use the fact that a_n is multiplicative in n to rewrite the sum in question in the form

$$\sum_{\substack{(x_0 - m_1)/d < n \leq t/d \\ (n, m) = 1}} \mu^2(dn) a_{dn} \chi(n) n^{-iT} = \mu^2(d) a_d \sum_{(x_0 - m_1)/d < n \leq t/d} \mu^2(n) a_n \chi_1(n) n^{-iT},$$

where $\chi_1(n) = \chi(n) \chi_0(n)$, $\chi_0(n)$ being the principal character modulo dm . Similarly as in section 6, we relate the sum over n on the right-hand side to the corresponding Dirichlet series, which we write in the form

$$\sum_{n=1}^{\infty} \mu^2(n) a_n \chi_1(n) n^{-s} = \prod_p (1 + a_p \chi_1(p) p^{-s}) = H(s, \chi_1) F(s, \chi_1),$$

where

$$H(s, \chi_1) = \prod_p \frac{1 + a_p \chi_1(p) p^{-s}}{\sum_{k=0}^{\infty} a_{p^k} \chi_1^k(p) p^{-ks}} = \prod_p \left(1 - \frac{\sum_{k=2}^{\infty} a_{p^k} \chi_1^k(p) p^{-ks}}{\sum_{k=0}^{\infty} a_{p^k} \chi_1^k(p) p^{-ks}} \right).$$

By $a_n \ll n^\varepsilon$ and (7.1), the product on the right-hand side converges absolutely and uniformly in every compact subset S of the half plane $\Re s > 1/2$. Hence, the function $H(s, \chi_1)$ is entire there. Moreover, $|H(s, \chi_1)|$ is bounded by a constant $C(\varepsilon)$ if $\Re s \geq 1/2 + \varepsilon$. The rest of the proof follows the arguments in the proof of Theorem 1, where the function $G_d(s, \chi)$ is replaced by $H(s, \chi_1)$, and in the application of Cauchy's integral theorem, the line of integration is shifted to $\Re s = 1/2 + \varepsilon$ instead of $\Re s = 1/2$.

10 Proof of Theorem 3

The general procedure of the proof will be similar as in [1], where we bounded the sum

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda([n^c]).$$

Therefore, we will be very brief in general and go into details only in the parts where the proof of Theorem 3 deviates substantially from that of Theorem 1 in [1]. First, we use the well-known relation

$$\lambda(p)^2 = 1 + \lambda(p^2).$$

Hence, we have

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda([n^c])^2 = \sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} 1 + \sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda([n^c]^2). \quad (10.1)$$

The ordinary Piatetski-Shapiro prime number theorem (see [1], for example) tells us that

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} 1 \sim \frac{N}{c \log N} \quad \text{as } N \rightarrow \infty$$

for every fixed c in the range in Theorem 3. It remains to estimate the second sum on the right-hand side of (10.1). We write this sum in the form

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda([n^c]^2) = \sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} b_{[n^c]},$$

where

$$b_n := \mu^2(n) \lambda(n^2) = \mu^2(n) a_n, \quad (10.2)$$

with a_n as in (2.3).

Clearly, it now suffices to bound the sum

$$\sum_{n \leq N} b_{[n^c]} \Lambda([n^c]),$$

where $\Lambda(n)$ is the von Mangoldt function. Similarly as in [1], we pull out a main term which we estimate using an analogue of the prime number theorem for $\lambda(p^2)$. Then we reduce the error

term to exponential sums as in [1] and treat the von Mangoldt function appearing in them using a Vaughan-type identity due to Heath-Brown (Lemma 4 in [1]). This leads to type I and type II sums. In [1], we then used the decomposition in (2.2) to separate the summation variables m and n in the said type I and type II sums. The decomposition of b_{mn} needed here is simpler since we have

$$b_{mn} = \begin{cases} 0 & \text{if } (m, n) > 1, \\ b_m b_n & \text{if } (m, n) = 1 \end{cases}$$

due to the appearance of the Möbius function in the definition of b_n . Now the type I and type II sums take the form

$$K = \sum_{h \sim H} \sum_{\substack{m \sim X \\ (m, n) = 1}} \sum_{n \sim Y} C_h A_m b_n e(h(mn)^\gamma) \quad (10.3)$$

and

$$L = \sum_{h \sim H} \sum_{\substack{m \sim X \\ (m, n) = 1}} \sum_{n \sim Y} C_h A_m B_n e(h(mn)^\gamma),$$

where

$$\gamma = \frac{1}{c}, \quad 1 \leq H \leq N^{1-\gamma+\eta} \quad \text{and} \quad XY = N. \quad (10.4)$$

Here C_h , A_m and B_n are general coefficients of size $\ll N^\varepsilon$, and b_n is defined as in (10.2).

We remove the coprimality condition $(m, n) = 1$ in K and L using Möbius inversion, getting

$$K = \sum_d \mu(d) K_d \quad \text{and} \quad L = \sum_d \mu(d) L_d \quad (10.5)$$

with

$$K_d = \sum_{h \sim H} \sum_{m \sim X/d} \sum_{n \sim Y/d} C_h A_{dm} b_{dn} e(h(d^2 mn)^\gamma).$$

and

$$L_d = \sum_{h \sim H} \sum_{m \sim X/d} \sum_{n \sim Y/d} C_h A_{dm} B_{dn} e(h(d^2 mn)^\gamma).$$

Using (10.5) and Lemmas 15, 16 and 18 in [1], we deduce that

$$L \ll N^{1-\eta} \quad \text{if } N^{1-\gamma+100\eta} \leq Y \leq N^{5\gamma-4-100\eta}$$

and

$$K \ll N^{1-\eta} \quad \text{if } N^{3-3\gamma+100\eta} \leq Y \leq N^{\gamma-100\eta}, \quad (10.6)$$

for some small $\eta > 0$, provided that $\gamma > 7/8$.

We note that if $0 < \gamma < 1$, then the function

$$f(x) = h(mx)^\gamma \quad (10.7)$$

satisfies the conditions on f in section 1, and if $\gamma > 1/2$, $h \sim H$, $m \sim X$, $Y \geq N^{2/3+100\eta}$, η is sufficiently small and (10.4) is satisfied, then condition (2.1), with N replaced by Y , in Theorems 1 and 2 holds, i.e.

$$Y^{1/2+\eta} \leq f(Y) \leq Y^{3/2-\eta}.$$

Now applying Theorem 2 with $f(n)$ defined as above and a_n defined as in (2.3) to the inner sum over n on the right-hand side of (10.3), we get

$$\sum_{\substack{n \sim Y \\ (m,n)=1}} b_n e(h(mn)^\gamma) = \sum_{\substack{n \sim Y \\ (m,n)=1}} \mu^2(n) a_n e(h(mn)^\gamma) \ll H^{1/11} X^{\gamma/11} Y^{19/22+\gamma/11+\varepsilon}$$

and hence

$$K \ll H^{12/11} X^{1+\gamma/11} Y^{19/22+\gamma/11+\varepsilon},$$

provided that $1/2 < \gamma < 1$ and $Y \geq N^{2/3+100\eta}$. By (10.4), it follows that

$$K \ll N^{1-\eta} \quad \text{if } Y \geq N^{8-22\gamma/3+100\eta}, \quad (10.8)$$

provided that $1/2 < \gamma < 1$. The Y -ranges in (10.6) and (10.8) overlap if $\gamma > 24/25$ and η is small enough. Hence, we have

$$K \ll N^{1-\eta} \quad \text{if } Y \geq N^{3-3\gamma+100\eta},$$

provided that $24/25 < \gamma < 1$ and η is sufficiently small. The rest of the proof is similar as in section 12 in [1]. We note that the range $1 < c < 25/24$ in Theorem 3 comes from the above condition $24/25 < \gamma < 1$.

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